

# CONGRUENCE OBSTRUCTIONS TO PSEUDOMODULARITY OF FRICKE GROUPS

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**ABSTRACT.** A pseudomodular group is a finite coarea nonarithmetic Fuchsian group whose cusp set is exactly  $\mathbb{P}^1(\mathbb{Q})$ . Long and Reid constructed finitely many of these by considering Fricke groups, i.e., those that uniformize one-cusped tori. We prove that a zonal Fricke group with rational cusps is pseudomodular if and only if its cusp set is dense in the finite adeles of  $\mathbb{Q}$ . We then deduce that infinitely many such Fricke groups are not pseudomodular.

## 1. INTRODUCTION

A *cusp* of a Fuchsian group  $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$  is an  $x \in \mathbb{P}^1(\mathbb{R})$  that is the unique fixed point of an element of  $\Gamma$  (see [S] Ch. 1). The modular group  $\mathrm{PSL}_2(\mathbb{Z})$  is a finite coarea Fuchsian group whose cusps are  $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ . In [LR], Long and Reid show that there exist finite coarea Fuchsian subgroups of  $\mathrm{PSL}_2(\mathbb{Q})$  that are *not* commensurable with  $\mathrm{PSL}_2(\mathbb{Z})$  (i.e., not arithmetic) and whose cusp set equals  $\mathbb{P}^1(\mathbb{Q})$ . They call such groups *pseudomodular*.

Long and Reid studied a particular family  $\Delta(u^2, 2t)$  of Fricke groups as candidates for pseudomodularity. Fricke groups uniformize one-cusped hyperbolic tori and all such tori are uniformized by Fricke groups; see [A] and [K]. Among Long and Reid's stated open problems in [LR] is the determination of the values  $(u^2, 2t) \in \mathbb{Q} \times \mathbb{Q}$  for which  $\Delta(u^2, 2t)$  is pseudomodular. Recall that if  $\hat{\mathbb{Z}}$  is the profinite completion of  $\mathbb{Z}$ , then  $\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$  is an additive topological group having a basis of open neighborhoods of 0 consisting of  $m\hat{\mathbb{Z}}$  for  $m \in \mathbb{Q}$ . Then our chief result is:

**Theorem 1.1.** *The Fuchsian group  $\Delta(u^2, 2t)$  is pseudomodular or arithmetic if and only if its finite cusps are dense in the ring  $\mathbb{A}_{\mathbb{Q},f}$  of finite adeles over  $\mathbb{Q}$ .*

Theorem 1.1 is in fact true for arbitrary zonal Fuchsian subgroups of  $\mathrm{PSL}_2(\mathbb{Q})$ . We remark that for a given  $t$  only finitely many  $u^2$  yield arithmetic groups. We shall use refinements of Theorem 1.1 to give explicit, infinite families of  $\Delta(u^2, 2t)$  whose cusps are proper subsets of  $\mathbb{P}^1(\mathbb{Q})$ . For example:

**Theorem 1.2.** *Let  $p$  be a prime and  $t$  an integer at least 2. Then  $\Delta(p^{-2}, 2t)$  is neither pseudomodular nor arithmetic. In particular, infinitely many  $\Delta(u^2, 2t)$  are neither pseudomodular nor arithmetic.*

Our method is to find  $\Delta(u^2, 2t)$ -invariant subsets of  $\mathbb{Q}$  that are defined number-theoretically, either adelically or  $p$ -adically. In [LR], Long and Reid exhibit finitely many  $\Delta(u^2, 2t)$  that are neither pseudomodular nor arithmetic. For each such group, they provide a rational number fixed by a hyperbolic element of  $\Delta(u^2, 2t)$ .

Such fixed points cannot also be cusps; see [B], p. 199. We do not know whether rational hyperbolic fixed points exist for all non-pseudomodular  $\Delta$ , and in any case, our proofs do not require or produce them. So far, our results disprove pseudomodularity with a finite amount of data, in terms of either finitely many primes or (finite-index) congruence subgroups of the modular group  $\mathrm{PSL}_2(\mathbb{Z})$ . In the final section of this paper we discuss some further questions and possible generalizations.

## 2. RESULTS

The numerator and denominator of a rational number will be taken coprime with the denominator positive. Let  $v_p : \mathbb{Q}^\times \rightarrow \mathbb{Z}$  be the usual discrete valuation at prime  $p$ . If  $p$  does not divide the denominator of a rational number  $x$  (i.e.,  $v_p(x) \geq 0$ ) then say  $x$  is *integral* at  $p$ . Integral  $x$  have representatives in the ring of  $p$ -adic integers  $\mathbb{Z}_p$  and we write  $x \equiv m \pmod{p}$  to mean that the residue of  $x \bmod p\mathbb{Z}_p$  is  $m$  in  $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ . Finally, for nonzero rational  $x$ , we say  $p \mid x$  or  $p$  *divides*  $x$  when  $v_p(x) > 0$ .

Denote by  $\mathcal{C}_\infty(G)$  the cusp set of a Fuchsian group  $G$  and let the finite cusps be denoted by  $\mathcal{C}(G) := \mathcal{C}_\infty(G) \setminus \{\infty\}$ . For rationals  $u^2$  and  $t$  with  $0 < u^2 < t - 1$ , let  $\Delta(u^2, 2t)$  be the subgroup of  $\mathrm{PSL}_2(\mathbb{R})$  generated by the hyperbolic elements

$$g_1 = \frac{1}{\sqrt{-1+t-u^2}} \begin{pmatrix} t-1 & u^2 \\ 1 & 1 \end{pmatrix}, \quad g_2 = \frac{1}{u\sqrt{-1+t-u^2}} \begin{pmatrix} u^2 & u^2 \\ 1 & t-u^2 \end{pmatrix}.$$

As in [LR],  $\Delta$  is a zonal Fricke group freely generated by  $g_1$  and  $g_2$  with  $\mathcal{C}_\infty(\Delta)$  a nonempty subset of  $\mathbb{P}^1(\mathbb{Q})$ . In fact, by considering the traces of  $g_1$ ,  $g_2$  and  $g_1g_2$  and using results in §1 of [CS] on varieties of group representations, we can show that every Fricke group with cusps in  $\mathbb{P}^1(\mathbb{Q})$  is conjugate in  $\mathrm{PGL}_2(\mathbb{Q})$  to some  $\Delta(u^2, 2t)$ . Therefore, we can study all such Fricke groups by looking at the groups  $\Delta(u^2, 2t)$ . We are interested in when  $\mathcal{C}_\infty(\Delta) = \mathbb{P}^1(\mathbb{Q})$  or, equivalently, when  $\mathcal{C}(\Delta) = \mathbb{Q}$ . Let  $\Lambda(u^2, 2t)$  be the kernel of the homomorphism  $\Delta(u^2, 2t) \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$  given by  $g_1 \mapsto (1, 0)$  and  $g_2 \mapsto (0, 1)$ . Then  $\Lambda(u^2, 2t) \subset \mathrm{PSL}_2(\mathbb{Q})$  and  $\mathcal{C}(\Lambda) = \mathcal{C}(\Delta)$ . If  $\mathcal{C}(\Lambda)$  is not dense in  $\mathbb{A}_{\mathbb{Q},f}$ , or in some finite product  $\prod_i \mathbb{Q}_{p_i}$  with  $\mathbb{Q}$  embedded diagonally, then  $\mathcal{C}(\Lambda) \neq \mathbb{Q}$  since  $\mathbb{Q}$  is dense in each of those sets.

Given this obstruction to pseudomodularity, we ask three questions: (i) when are the finite cusps of  $\Delta(u^2, 2t)$  dense in a specified finite product  $\prod_i \mathbb{Q}_{p_i}$ , (ii) is cusp density in all such products a sufficient condition for pseudomodularity, and (iii) how do the answers to the former two questions vary as  $u^2$  and  $t$  are themselves varied  $p$ -adically? The propositions below address these questions and, in particular, prove Theorem 1.2.

**Proposition 2.1.** *Let  $p$  be prime. If  $v_p(t) \geq 0$  and  $v_p(u^2) \leq -2$ , or if  $v_p(t) < 0$  and  $v_p(u^2) \leq 2(v_p(t) - 1)$ , then  $\mathcal{C}(\Delta(u^2, 2t))$  is not dense in  $\mathbb{Q}_p$ .*

Considering conditions at two primes, we can show:

**Proposition 2.2.** *If  $p$  and  $q$  are prime,  $v_p(u^2) = -1 = v_q(u^2)$ , and  $t$  is integral at  $p$  and at  $q$ , then  $\mathcal{C}(\Delta(u^2, 2t))$  is not dense in  $\mathbb{Q}_p \times \mathbb{Q}_q$ .*

Results similar to Proposition 2.2 hold for  $t$  that are non-integral at  $p$  or at  $q$ ; we omit their statements for brevity. As a corollary to the above two propositions, whenever  $t$  is an integer and the denominator of  $u^2$  is composite,  $\Delta(u^2, 2t)$  is not pseudomodular. We have an even stronger result for integral  $t$ :

**Proposition 2.3.** *Let  $t$  be an integer and suppose  $\Delta(u^2, 2t)$  is pseudomodular. Then*

- (a)  $u^2$  has prime or unit denominator, say  $p$ ,
- (b) if this  $p$  is an odd prime, then  $p$  does not divide  $t$ , and
- (c) for all odd primes  $q$  dividing  $t$ ,  $u^2$  (necessarily in  $\mathbb{Z}_q$ ) is equivalent to 0 or  $-1 \pmod q$ .

To prove each of the above propositions, we give a proper nonempty  $\Delta$ -invariant subset  $U$  of  $\mathbb{Q}$  that is open in the topology induced by that of  $\mathbb{Q}_p$  or  $\mathbb{Q}_p \times \mathbb{Q}_q$ . For example, under the hypotheses of Proposition 2.2, the set of nonzero rationals  $x$  for which exactly one of  $v_p(x)$  and  $v_q(x)$  is negative is  $\Delta$ -invariant. To prove that our  $U$  are  $\Delta$ -invariant, we show that the generators  $g_i^{\pm 1}$  of  $\Delta$  map  $U$  into itself.

The difficulty of our approach lies in finding such sets  $U$ . To do this, we search computed data for patterns in valuations and congruences. We build a data set by taking a tuple  $(x_1, \dots, x_m)$  of rational numbers and applying to it several elements of  $\Delta$ . Useful values for the  $x_i$  are rational hyperbolic (or *special*) fixed points, such as those tabulated in [LR]. In some cases, a tuple of length one yields some information, but typically we let the tuple be given by multiple pairs of special fixed points, with each pair fixed by a single hyperbolic element of  $\Delta$ . When we have no data on special fixed points, we select the  $x_i$  with particular valuation or congruence properties. In any case, a successful search suggests a candidate  $U$ , which we then confirm as described above. Varying parameters such as the denominator of  $u^2$ , in such a way as to preserve the proof of the  $\Delta$ -invariance, allows us to eliminate infinite families of candidates for pseudomodularity and divorce our results from computed data.

In the other direction, to show that  $\mathcal{C}(\Delta)$  is dense in a finite product  $H = \prod_i \mathbb{Q}_{p_i}$ , we show that for each rational  $x$ , we can move  $x$  arbitrarily close to the cusp 0 with elements of (a group commensurable with)  $\Delta$ . We use this argument to prove that for  $t$  a prime integer, we have identified above *all* cases for which  $\mathcal{C}(\Delta(u^2, 2t))$  is not dense in some  $H$ :

**Proposition 2.4.** *If  $t$  is prime,  $u^2$  has prime denominator not equal to  $t$  and  $u^2 \equiv 0$  or  $-1 \pmod t$ , then  $\mathcal{C}(\Delta(u^2, 2t))$  is dense in every finite product  $\prod_i \mathbb{Q}_{p_i}$  and hence is dense in the product  $\prod_p \mathbb{Q}_p$  over all primes.*

There are groups with special fixed points to which this proposition applies, such as  $\Delta(6/11, 6)$  with a special fixed point of  $1/4$ . Consequently:

**Corollary 2.5.** *Density of  $\mathcal{C}(\Delta(u^2, 2t))$  in the product  $\prod_p \mathbb{Q}_p$  of all  $p$ -adic fields is not a sufficient condition for pseudomodularity.*

Both this corollary and Theorem 1.1 can be interpreted in terms of congruence data with respect to all primes. While density in the adelic topology implies pseudomodularity, density in the product topology on all  $p$ -adic fields does not. Theorem 1.1 holds because an orbit of a rational under a fixed translation  $x \mapsto x + t$  with  $t \in \mathbb{Q}$  is open in  $\mathbb{Q}$  in the topology induced by  $\mathbb{A}_{\mathbb{Q},f}$ .

### 3. QUESTIONS

Since  $\Lambda(u^2, 2t)$  is a finitely generated subgroup of  $\mathrm{PSL}_2(\mathbb{Q})$ , it is always a subgroup of  $\mathrm{PSL}_2(\mathbb{Z}_S)$ , where  $\mathbb{Z}_S = \mathbb{Z}[p_1^{-1}, \dots, p_r^{-1}]$  for some minimal set of primes  $p_1, \dots, p_r$ . By Riemann-Hurwitz,  $\Lambda(u^2, 2t)$  always has exactly four orbits of cusps,

so if we can show that  $\Lambda(u^2, 2t)$  has more than four orbits in its action on  $\mathbb{P}^1(\mathbb{Q})$ , then some rationals cannot be cusps, whence  $\Lambda(u^2, 2t)$  is not pseudomodular.

**Question 1.** For a non-pseudomodular  $\Lambda(u^2, 2t) \subset \mathrm{PSL}_2(\mathbb{Z}_S)$  as above, can we find a subgroup  $K$  of (finite index in)  $\mathrm{PSL}_2(\mathbb{Z}_S)$  such that  $\Lambda(u^2, 2t) \subset K$  and  $K$  has more than four orbits in its action on  $\mathbb{P}^1(\mathbb{Q})$ ?

For the groups already eliminated by our earlier propositions, the answer is yes. We also have a positive answer for some groups whose cusps are dense in the product of all  $\mathbb{Q}_p$ 's, such as  $\Lambda(6/11, 6)$ . In fact, we can construct a  $K \subset \mathrm{PSL}_2(\mathbb{Z}[2^{-1}])$  containing  $\Lambda(6/11, 6)$  that has eight orbits in its action on  $\mathbb{P}^1(\mathbb{Q})$ . An explicit description of the orbits of  $K$  gives us a  $\Delta(6/11, 6)$ -invariant, nonempty, proper subset  $X$  of  $\mathbb{P}^1(\mathbb{Q})$  that is open in the 33-congruence topology, as defined here.

**Definition 3.1.** Denote by  $\Gamma(M)$  the principal congruence subgroup of  $\mathrm{PSL}_2(\mathbb{Z})$  of level  $M$ . For a finite set of primes  $p_1, \dots, p_r$ , let  $N = p_1 \dots p_r$  and let  $\mathcal{V}_N$  be the set of orbits  $\Gamma(N^j) \cdot 0$  as  $j$  ranges over positive integers. The  $N$ -congruence topology on  $\mathbb{P}^1(\mathbb{Q})$  is the topology generated by the  $\mathrm{PSL}_2(\mathbb{Z})$ -translates of  $\mathcal{V}_N$ .

If we replace  $\Gamma(N^j)$  with  $\Gamma^0(N^j)$  (matrices lower-triangular mod  $N^j$ ) in this definition then we recover the (coarser) topology on  $\mathbb{P}^1(\mathbb{Q})$  induced by inclusion in  $\prod_{i=1}^r \mathbb{P}^1(\mathbb{Q}_{p_i})$ . Thus, in Propositions 2.1 and 2.2, we also have cusp sets failing to be dense in some congruence topology.

**Question 2.** If the cusps of  $\Delta$  are dense in  $\mathbb{P}^1(\mathbb{Q})$  in all congruence topologies, is  $\Delta$  necessarily pseudomodular or arithmetic? Equivalently, if  $\Delta$  is neither pseudomodular nor arithmetic, is there a congruence subgroup  $\Gamma$  of  $\mathrm{PSL}_2(\mathbb{Z})$  so that  $\mathcal{C}(\Delta)$  misses some orbit  $\Gamma \cdot x \in \mathbb{P}^1(\mathbb{Q})$ ?

More generally, we can allow non-congruence subgroups  $\Gamma$  and we can ask analogous questions for larger families of Fuchsian groups or for Kleinian groups whose cusps lie in an imaginary quadratic number field.

## REFERENCES

- [A] R. Abe, On Correspondences between once punctured tori and closed tori: Fricke groups and real lattices, *Tokyo J. Math.* 23 (2000), no. 2, 269–293.
- [B] A. F. Beardon, *The Geometry of Discrete Groups*, Grad. Texts in Math., vol. 91., Springer-Verlag, 1983.
- [CS] C. Culler, P. Shalen, Varieties of group representations and splittings of 3-manifolds, *Annals of Math.* 117 (1983), no. 1, 109–146.
- [K] L. Keen, On fundamental domains and the Teichmüller modular group, *Contributions to Analysis*, Academic Press (1974), 185–194.
- [LR] D. D. Long, A. W. Reid, Pseudomodular surfaces, *J. Reine Angew. Math.* 552 (2002) 77–100.
- [S] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, 1971.

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